Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 95412, 9 pages
doi:10.1155/2007/95412

Research Article

A General Iterative Method for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

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Received 14 May 2007; Revised 15 August 2007; Accepted 18 September 2007

Recommended by Hichem Ben-El-Mechaiekh

We introduce a general iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space. Our results improve and extend the corresponding ones announced by S. Takahashi and W. Takahashi in 2007, Marino and Xu in 2006, Combettes and Hirstoaga in 2005, and many others.

1. Introduction

Let H be a real Hilbert space and let C be nonempty closed convex subset of H. Recall that a mapping S of C into itself is called nonexpansive if $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(S)$ the set of fixed points of S. Let B be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $B : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$B(x, y) \geq 0 \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(B)$. Give a mapping $T : C \to H$, let $B(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(B)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$, that is, z is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1, 2]. Recently, Combettes and Hirstoaga [1] introduced an iterative scheme of finding the best approximation to the initial data when $EP(B)$ is nonempty and proved a strong convergence theorem. Very recently, S. Takahashi and W. Takahashi [3] also introduced a new iterative scheme:

$$B(y_n,u)+\frac{1}{r_n}\langle u-y_n,y_n-x_n\rangle\geq 0,\quad \forall u\in C,$$
$$x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)Sy_n, \tag{1.2}$$

for approximating a common element of the set of fixed points of a nonself nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space.

Recall that a linear bounded operator A is strongly positive if there is a constant $\overline{\gamma}>0$ with property $\langle Ax,x\rangle\geq\overline{\gamma}\|x\|^2$, $\forall x\in H$.

Recently iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, for example, [4–7] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.3}$$

where C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [6], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0\in H$ chosen arbitrarily, $x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_nb$, $n\geq 0$, converges strongly to the unique solution of the minimization problem (1.3) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [8] introduced a new iterative scheme by the viscosity approximation method [9]:

$$x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_n\gamma f(x_n),\qquad n\geq 0. \tag{1.4}$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality $\langle(A-\gamma f)x^*,x-x^*\rangle\geq 0, x\in C$, which is the optimality condition for the problem $\min_{x\in C}(1/2)\langle Ax,x\rangle-h(x)$, where C is the fixed point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

In this paper, motivated by Combettes and Hirstoaga [1], Moudafi [9], S. Takahashi and W. Takahashi [3], Marino and Xu [8], and Wittmann [10], we introduce a general iterative scheme as following:

$$B(y_n,u)+\frac{1}{r_n}\langle u-y_n,y_n-x_n\rangle\geq 0,\quad \forall\, u\in C,$$
$$x_{n+1}=\alpha_n\gamma f(x_n)+(I-\alpha_nA)Sy_n. \tag{1.5}$$

We will prove that the sequence $\{x_n\}$ generated by (1.5) converges strongly to a common element of the set of fixed points of nonexpansive mapping S and the set of solutions of equilibrium problem (1.1), which is the unique solution of the variational inequality $\langle\gamma f(q)-Aq,q-p\rangle\leq 0$, $\forall\, p\in F$, where $F=F(S)\cap EP(B)$ and is also the optimality condition for the minimization problem $\min_{x\in F}(1/2)\langle Ax,x\rangle-h(x)$, where h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. It is well known that for all $x,y\in H$ and $\lambda\in[0,1]$, there holds

$$||\lambda x+(1-\lambda)y||^2=\lambda\|x\|^2+(1-\lambda)\|y\|^2-\lambda(1-\lambda)\|x-y\|^2. \tag{2.1}$$

A space X is said to satisfy Opial's condition [11] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x\in X$, we have

$$\liminf_{n\to\infty}||x_n-x||<\liminf_{n\to\infty}||x_n-y||,\quad \forall y\in X,\ y\neq x. \tag{2.2}$$

For solving the equilibrium problem for a bifunction $B: C\times C\to\mathbb{R}$, let us assume that B satisfies the following conditions:

(A1) $B(x,x)=0$ for all $x\in C$;

(A2) B is monotone, that is, $B(x,y)+B(y,x)\leq 0$ for all $x,y\in C$;

(A3) for each $x,y,z\in C$, $\lim_{t\downarrow 0}B(tz+(1-t)x,y)\leq B(x,y)$;.

(A4) for each $x\in C$, $y\mapsto B(x,y)$ is convex and lower semicontinuous.

Lemma 2.1 [5]. *Assume $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1}\leq(1-\gamma_n)\alpha_n+\delta_n,\quad n\geq 0, \tag{2.3}$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty}\gamma_n=\infty$;

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n\leq 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

Then $\lim_{n\to\infty}\alpha_n=0$.

Lemma 2.2 [12]. *Let C be a nonempty closed convex subset of H and let B be a bifunction of $C\times C$ into $\mathbb{R}$ satisfying* (A1)–(A4). *Let $r>0$ and $x\in H$. Then, there exists $z\in C$ such that $B(z,y)+(1/r)\langle y-z,z-x\rangle\geq 0$, $\forall y\in C$.*

Lemma 2.3 [1]. *Assume that $B: C\times C\to\mathbf{R}$ satisfies (A1)–(A4). For $r>0$ and $x\in H$, define a mapping $T_r: H\to C$ as follows:*

$$T_r(x)=\left\{z\in C:\ B(z,y)+\frac{1}{r}\langle y-z,z-x\rangle\geq 0,\ \forall y\in C\right\} \tag{2.4}$$

for all $z\in H$. Then, the following hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x,y\in H$,*

$$||T_rx-T_ry||^2\leq\langle T_rx-T_ry,x-y\rangle; \tag{2.5}$$

(3) *$F(T_r)=EP(B)$;*

(4) *$EP(B)$ is closed and convex.*

Lemma 2.4. *In a real Hilbert space H, there holds the the inequality $\|x+y\|^2\leq\|x\|^2+2\langle y,x+y\rangle$, for all $x,y\in H$.*

Lemma 2.5 [8]. *Assume that A is a strong positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$. Then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H. Let B be a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4) and let S be a nonexpansive mapping of C into H such that $F(S) \cap EP(B) \neq \varnothing$ and a strongly positive linear bounded operator A with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$B(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \ge 0, \quad \forall u \in C,$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A) S y_n \tag{3.1}$$

for all n, where $\{\alpha_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy
(C1) $\lim_{n\to\infty} \alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C4) $\liminf_{n\to\infty} r_n > 0$.
Then, both $\{x_n\}$ and $\{y_n\}$ converge strongly to $q \in F(S) \cap EP(B)$, where $q = P_{F(S)\cap EP(B)}(\gamma f + (I - A))(q)$, which solves some variation inequality:

$$\langle \gamma f(q) - Aq, q - p\rangle \le 0, \quad \forall p \in F(S) \cap EP(B). \tag{3.2}$$

Proof. Since $\alpha_n \to 0$ by the condition (C1), we may assume, with no loss of generality, that $\alpha_n < \|A\|^{-1}$ for all n. From Lemma 2.5, we know that if $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$. We will assume that $\|I - A\| \le 1 - \overline{\gamma}$.

Now, we observe that $\{x_n\}$ is bounded. Indeed, pick $p \in F(S) \cap EP(B)$. Since $y_n = T_{r_n} x_n$, we have

$$||y_n - p|| = ||T_{r_n} x_n - T_{r_n} p|| \le ||x_n - p||. \tag{3.3}$$

It follows that

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(S y_n - p)|| \\ &\le [1 - (\overline{\gamma} - \gamma\alpha)\alpha_n]||x_n - p|| + \alpha_n||\gamma f(p) - Ap||, \end{aligned} \tag{3.4}$$

which gives that $\|x_n - p\| \le \max\{\|x_0 - p\|, \|\gamma f(p) - Ap\|/(\overline{\gamma} - \gamma\alpha)\}$, $n \ge 0$. Therefore, we obtain that $\{x_n\}$ is bounded. So is $\{y_n\}$. Next, we show that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.5}$$

Observing that $y_n = T_{r_n}x_n$ and $y_{n+1} = T_{r_{n+1}}x_{n+1}$, we have

$$B(y_n,u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, \quad \forall\, u \in C, \tag{3.6}$$

$$B(y_{n+1},u) + \frac{1}{r_{n+1}}\langle u - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0, \quad \forall\, u \in C. \tag{3.7}$$

Putting $u = y_{n+1}$ in (3.6) and $u = y_n$ in (3.7), we have

$$B(y_n, y_{n+1}) + \frac{1}{r_n}\langle y_{n+1} - y_n, y_n - x_n\rangle \geq 0 \tag{3.8}$$

and $B(y_{n+1}, y_n) + (1/r_{n+1})\langle y_n - y_{n+1}, y_{n+1} - x_{n+1}\rangle \geq 0$. It follows from (A2) that

$$\left\langle y_{n+1} - y_n, \frac{y_n - x_n}{r_n} - \frac{y_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0. \tag{3.9}$$

That is, $\langle y_{n+1} - y_n, y_n - y_{n+1} + y_{n+1} - x_n - (r_n/r_{n+1})(y_{n+1} - x_{n+1})\rangle \geq 0$. Without loss of generality, let us assume that there exists a real number m such that $r_n > m > 0$ for all n. It follows that

$$||y_{n+1} - y_n||^2 \leq ||y_{n+1} - y_n||\left(||x_{n+1} - x_n|| + \left|1 - \frac{r_n}{r_{n+1}}\right| ||y_{n+1} - x_{n+1}||\right). \tag{3.10}$$

It follows that

$$||y_{n+1} - y_n|| \leq ||x_{n+1} - x_n|| + M_1\,|\,r_{n+1} - r_n\,|, \tag{3.11}$$

where M_1 is an appropriate constant such that $M_1 \geq \sup_{n\geq 1}\|y_n - x_n\|$. Observe that

$$\begin{aligned} ||x_{n+2} - x_{n+1}|| \leq{}& (1 - \alpha_{n+1}\overline{\gamma})||y_{n+1} - y_n|| + |\,\alpha_{n+1} - \alpha_n\,|\,||ASy_n|| \\ &+ \gamma[\alpha_{n+1}\alpha||x_{n+1} - x_n|| + |\,\alpha_{n+1} - \alpha_n\,|\,||f(x_n)||]. \end{aligned} \tag{3.12}$$

Substitute (3.11) into (3.12) yields that

$$||x_{n+2} - x_{n+1}|| \leq [1 - (\overline{\gamma} - \gamma\alpha)\alpha_{n+1}]||x_{n+1} - x_n|| + M_2(2\,|\,\alpha_{n+1} - \alpha_n\,| + |\,r_{n+1} - r_n\,|), \tag{3.13}$$

where M_2 is an appropriate constant. An application of Lemma 2.1 to (3.13) implies that

$$\lim_{n\to\infty}||x_{n+1} - x_n|| = 0. \tag{3.14}$$

Observing (3.11), (3.14), and condition (C3), we have

$$\lim_{n\to\infty}||y_{n+1} - y_n|| = 0. \tag{3.15}$$

Since $x_n = \alpha_{n-1}\gamma f(x_{n-1}) + (I - \alpha_{n-1}A)Sy_{n-1}$, we have

$$||x_n - Sy_n|| \leq \alpha_{n-1}||\gamma f(x_n) - ASy_{n-1}|| + ||y_{n-1} - y_n||, \tag{3.16}$$

which combines with $\alpha_n \to 0$, and (3.15) gives that

$$\lim_{n\to\infty} ||x_n - Sy_n|| = 0. \tag{3.17}$$

For $p \in F(S) \cap EP(B)$, we have

$$\begin{aligned} ||y_n - p||^2 &= ||T_{r_n}x_n - T_{r_n}p||^2 \leq \langle T_{r_n}x_n - T_{r_n}p, x_n - p\rangle = \langle y_n - p, x_n - p\rangle \\ &= \frac{1}{2}\left(||y_n - p||^2 + ||x_n - p||^2 - ||x_n - y_n||^2\right), \end{aligned} \tag{3.18}$$

and hence $\|y_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - y_n\|^2$. It follows that

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||\alpha_n(\gamma f(x_n) - Ap) + (I - \alpha_n A)(Sy_n - p)||^2 \\ &\leq \alpha_n||\gamma f(x_n) - Ap||^2 + ||x_n - p||^2 - (1 - \alpha_n\overline{\gamma})||x_n - y_n||^2 \\ &\quad + 2\alpha_n(1 - \alpha_n\overline{\gamma})||\gamma f(x_n) - Ap||\,||y_n - p||. \end{aligned} \tag{3.19}$$

That is,

$$\begin{aligned} (1 - \alpha_n\overline{\gamma})||x_n - y_n||^2 \leq \alpha_n&||\gamma f(x_n) - Ap||^2 \\ &+ (||x_n - p|| + ||x_{n+1} - p||)||x_n - x_{n+1}|| \\ &+ 2\alpha_n(1 - \alpha_n\overline{\gamma})||\gamma f(x_n) - Ap||\,||y_n - p||. \end{aligned} \tag{3.20}$$

It follows from $\lim_{n\to\infty}\alpha_n = 0$ that

$$\lim_{n\to\infty} ||x_n - y_n|| = 0. \tag{3.21}$$

Observe from $\|Sy_n - y_n\| \leq \|Sy_n - x_n\| + \|x_n - y_n\|$, which combines with (3.17) and (3.21), that

$$\lim_{n\to\infty} ||Sy_n - y_n|| = 0. \tag{3.22}$$

On the other hand, we have

$$||x_n - Sx_n|| = ||Sx_n - Sy_n|| + ||Sy_n - x_n|| \leq ||x_n - y_n|| + ||Sy_n - x_n||. \tag{3.23}$$

It follows from (3.17) and (3.21) that $\lim_{n\to\infty}\|Sx_n - x_n\| = 0$. Observe that $P_{F(S)\cap EP(B)}(\gamma f + (I - A))$ is a contraction. Indeed, $\forall x, y \in H$, we have

$$\begin{aligned} &||P_{F(S)\cap EP(B)}(\gamma f + (I - A))(x) - P_{F(S)\cap EP(B)}(\gamma f + (I - A))(y)|| \\ &\quad\leq \gamma||f(x) - f(y)|| + \|I - A\|\,\|x - y\| \\ &\quad\leq \gamma\alpha\|x - y\| + (1 - \overline{\gamma})\|x - y\| < \|x - y\|. \end{aligned} \tag{3.24}$$

Banach's contraction mapping principle guarantees that $P_{F(S)\cap EP(B)}(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_{F(S)\cap EP(B)}(\gamma f + (I - A))(q)$. Next, we show that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle \le 0. \tag{3.25}$$

To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle = \lim_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle. \tag{3.26}$$

Correspondingly, there exists a subsequence $\{y_{n_i}\}$ of $\{y_n\}$. Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $y_{n_i} harpoonupw$. From (3.22), we obtain $Sy_{n_i} harpoonupw$.

Next, we show $w \in F(S) \cap EP(B)$. First, we prove $w \in EP(B)$. Since $y_n = T_{r_n} x_n$, we have $B(y_n, u) + (1/r_n)\langle u - y_n, y_n - x_n\rangle \ge 0$ for all $u \in C$. It follows from (A2) that $\langle u - y_n, (y_n - x_n)/r_n\rangle \ge B(u, y_n)$. Since $(y_{n_i} - x_{n_i})/r_{n_i} \to 0$, $y_{n_i} harpoonupw$, and (A4), we have $B(u, w) \le 0$ for all $u \in C$. For t with $0 < t \le 1$ and $u \in C$, let $u_t = tu + (1 - t)w$. Since $u \in C$ and $w \in C$, we have $u_t \in C$ and hence $B(u_t, w) \le 0$. So, from (A1) and (A4), we have $0 = B(u_t, u_t) \le tB(u_t, u) + (1 - t)B(u_t, w) \le tB(u_t, u)$. That is, $B(u_t, u) \ge 0$. It follows from (A3) that $B(w, u) \ge 0$ for all $u \in C$ and hence $w \in EP(B)$. Since Hilbert spaces are *Opial's* spaces, from (3.22), we have

$$\liminf_{n\to\infty} ||y_{n_i} - w|| \le \liminf_{n\to\infty} ||Sy_{n_i} - Sw|| \le \liminf_{n\to\infty} ||y_{n_i} - w|| < \liminf_{n\to\infty} ||y_{n_i} - Sw||, \tag{3.27}$$

which derives a contradiction. Thus, we have $w \in F(S)$. That is, $w \in F(S) \cap EP(B)$. Since $q = P_{F(S)\cap EP(B)} f(q)$, we have

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle = \lim_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle = \langle \gamma f(q) - Aq, w - q\rangle \le 0. \tag{3.28}$$

That is, (3.25) holds. Next, it follows Lemma 2.4 that

$$\begin{aligned} &||x_{n+1} - q||^2 \\ &\quad \le (1 - \alpha_n\overline{\gamma})^2 ||x_n - q||^2 + \alpha_n\gamma\alpha\left(||x_n - q||^2 + ||x_{n+1} - q||^2\right) + 2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q\rangle, \end{aligned} \tag{3.29}$$

which implies that

$$\begin{aligned} ||x_{n+1} - q||^2 \le{}& \left[1 - \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\right]||x_n - q||^2 \\ &+ \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\left[\frac{1}{\overline{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle + \frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \alpha\gamma)}M_3\right], \end{aligned} \tag{3.30}$$

where M_3 is an appropriate constant such that $M_3 = \sup_{n\to\infty}\|x_n - q\|$ for all n. Put $l_n = 2\alpha_n(\overline{\gamma} - \alpha_n\gamma)/(1 - \alpha_n\alpha\gamma)$ and $t_n = (1/(\overline{\gamma} - \alpha\gamma))\langle\gamma f(q) - Aq, x_{n+1} - q\rangle + (\alpha_n\overline{\gamma}^2/2(\overline{\gamma} - \alpha\gamma))M_3$. That is,

$$||x_{n+1} - q||^2 \leq (1 - l_n)||x_n - q|| + l_n t_n. \tag{3.31}$$

It follows from condition (C1), (C2), and (3.25) that $\lim_{n\to\infty} l_n = 0, \sum_{n=1}^{\infty} l_n = \infty$, and $\limsup_{n\to\infty} t_n \leq 0$. Apply Lemma 2.1 to (3.31) to conclude $x_n \to q$. □

4. Applications

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H. and let S be a nonexpansive mapping of C into H such that $F(S) \neq \varnothing$. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α $(0 < \alpha < 1)$ and let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (I - \alpha_n A)SP_C x_n \tag{4.1}$$

for all n, where $\alpha_n \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$ satisfy
(C1) $\lim_{n\to\infty}\alpha_n = 0$;
(C2) $\sum_{n=1}^{\infty}\alpha_n = \infty$;
(C3) $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$.
Then $\{x_n\}$ converges strongly to $q \in F(S)$, where $q = P_{F(S)}(\gamma f + (I - A))(q)$.

Proof. Put $B(x,y) = 0$ for all $x, y \in C$ and $\{r_n\} = 1$ for all n in Theorem 3.1. Then we have $y_n = P_C x_n$. So, the sequence $\{x_n\}$ converges strongly to $q \in F(S)$, where $q = P_{F(S)}(\gamma f + (I - A))(q)$. □

Remark 4.2. It is very clear that our algorithm with a variational regularization parameter $\{r_n\}$ has certain advantages over the algorithm with a fixed regularization parameter r. In some setting, when the regularization parameter $\{r_n\}$ depends on the iterative step n, the algorithm may converge to some solution Q-superlinearly, that is, the algorithm has a faster convergence rate when the regularization parameter $\{r_n\}$ depends on n, see [13] and the references therein for more information.

Acknowledgment

This project is supported by the National Natural Science Foundation of China under Grant no. 10771050.

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